November 18, 2006
Bethel College

Blaise Pascal Mathematics Competition

Directions:

- Do not open this booklet until you are instructed to do so.
- This exam contains forty multiple choice questions, each with five possible answers. Only one answer is the correct answer.
- Please record your answers on the given **scantron sheet** using only spaces 1 through 45. Use a No. 2 pencil (either the one provided or your own) for marking your answers. Do not make any extraneous marks on your scantron sheet, as this will affect the scoring. Be sure that you completely fill in the bubble, and erase your mistakes fully.
- Scoring will be as follows: You will receive 1 point for a correct answer, zero points for an answer left blank, and $-\frac{1}{4}$ for an incorrect answer. Thus, it is not to your advantage to guess.
- You will have 90 minutes to complete the exam.
- If your booklet is defective or has a missing page, please notify someone immediately.
- Drawings on the exam are not to scale.
- No external aids may be utilized during the exam except a pencil and scratch paper. In particular, no calculators will be allowed during the exam.
- When time is called, you must stop working and turn in your scantron form. The paper copy of the exam is yours to keep, as is the pencil. You may also collect an answer sheet on your way out.

1. If $(x^6-1)(x^3+1) = (x^2-1)h(x)$, what is $h(1)$?

   (a) 0 (b) 5 (c) 6 (d) 15 (e) 18

2. Which of the following is the correct order of the real numbers

$$x = \frac{2005}{2006}, \quad y = 2005^{1/2006}, \quad z = \left(\frac{1}{2006}\right)^{2005}$$

   (a) $x < y < z$ (b) $y < z < x$ (c) $z < y < x$ (d) $x < z < y$ (e) $z < x < y$

3. Let $(a_n)_{n=1}^{\infty}$ be a sequence defined by $a_1 = 11$ and for $n \geq 1$,

$$a_{n+1} = \begin{cases} (a_n)/2 & \text{if } a_n \text{ even} \\ 3a_n + 1 & \text{if } a_n \text{ odd} \end{cases}$$

   What is a_{2006}?

   (a) 0 (b) 1 (c) 2 (d) 3 (e) 4

4. For $n \geq 3$, define the function

$$f(n) = \log_2 3 \cdot \log_3 4 \cdot \log_4 5 \cdot \cdots \cdot \log_{n-1} n$$

   What is the integer closest to $f(2006)$?

   (a) 10 (b) 11 (c) 12 (d) 13 (e) 14

5. If $(x^3+2x^2-3x+4)(x^4-2x^3+kx^2-2x+1) = x^7-22x^4-32x^3+34x^2-11x+4$, what is the value of k?

   (a) 5 (b) 6 (c) 7 (d) 8 (e) 9

6. How many points do the graphs of $x^2+36y^2 = 36$ and $x^2-y^2 = 4$ have in common?

   (a) 0 (b) 2 (c) 4 (d) 6 (e) 8

7. Suppose that the operation $\square$ is defined by $a \square b = a + b + ab$ (where a and b are integers). Which of the following is equivalent to $(a \square b) \square (-a)$?

   (a) $(ab) \square (-a)$ (b) $b \square (-a^2)$ (c) $-a^2 b$ (d) $a \square (-ab)$ (e) $b - a^2 b$

8. What is the last digit of 1118^{2006}?

   (a) 0 (b) 2 (c) 4 (d) 6 (e) 8

9. Suppose that $\overline{AB}$ is the diameter of the circle shown at right, and $\angle ABC = 75^\circ$. Find the length of the arc $\widehat{BC}$?

   (a) $\frac{2\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$ (e) $\frac{3\pi}{4}$

10. What is the remainder (in base ten) when the number 111010110101_2 is divided by 31_5? (Here, the subscript denotes the base in which the number is written.)

    (a) 1 (b) 3 (c) 5 (d) 7 (e) 9

11. Which of the following is equal to the sum

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{2005 \cdot 2006}$$

    (a) $\frac{2007}{2006}$ (b) $\frac{2006}{2005}$ (c) 1 (d) $\frac{2005}{2006}$ (e) none of these

12. What is the length of the longest rod that can be placed in a right circular cylinder with height 8 and base radius 4?

    (a) $4\sqrt{5}$ (b) $8\sqrt{2}$ (c) 12 (d) $16\sqrt{2}$ (e) $16\sqrt{3}$

13. The number $\dfrac{2006!}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot 2005}$ is equal to which of the following?

(a) $2^{1003}(1003!)$ (b) $2^{2006}(1003!)$ (c) 2006 (d) 1003! (e) $2^{1002}(1002!)$

14. Let $(a_n)_{n=1}^{\infty}$ be a sequence defined by $a_1 = 2006$ and for $n \geq 1$, $a_{n+1} = \sqrt{2 + a_n}$. As n approaches infinity, what value does a_n approach?

(a) -1 (b) 1 (c) $\sqrt{2}$ (d) 2 (e) $2\sqrt{2}$

15. A cube of side 3 is painted red on all six faces and then cut into individual cubes of side length 1. These cubes are then placed in a bag, and one cube is drawn randomly. What is the probability that the cube has at least two faces painted red?

(a) $\dfrac{2}{3}$ (b) $\dfrac{19}{27}$ (c) $\dfrac{20}{27}$ (d) $\dfrac{7}{9}$ (e) $\dfrac{22}{27}$

16. Suppose that the decimal expansion of $N!$ ends with five zeros. What is the smallest possible value of N?

(a) 20 (b) 22 (c) 23 (d) 25 (e) 26

17. $\arcsin \frac{1}{4} + \arccos \frac{1}{4} + \arctan \frac{1}{4} + \operatorname{arccot} \frac{1}{4} =$

(a) $\dfrac{\pi}{2}$ (b) $\dfrac{2\pi}{3}$ (c) π (d) $\dfrac{4\pi}{3}$ (e) 2π

18. Let $d(n)$ denote the sum of the digits of the number n. For example, $d(159) = 1+5+9 = 15$. Let $d^2(n) = d(d(n))$, $d^3(n) = d(d(d(n)))$, and so on. Find $d^{11182006}(11182006)$.

(a) 1 (b) 3 (c) 5 (d) 7 (e) 9

19. Find the area of a circle that circumscribes an equilateral triangle with side length 1.

(a) $\dfrac{\pi}{2}$ (b) $\dfrac{\pi}{3}$ (c) $\dfrac{\pi}{6}$ (d) $\dfrac{\pi}{9}$ (e) π

20. How many distinguishable ways can the letters in the word CONTEST be rearranged? (Note that switching the C and the O yields a distinguishable rearrangement, but simply switching the two T's does not.)

(a) 1260 (b) 2500 (c) 5040 (d) 720 (e) 2520

21. Suppose that a and b are the real roots of the equation $x^2 - 10x + 5 = 0$. What is the value of $\frac{1}{a} + \frac{1}{b}$?

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5

22. If $\sin x + \cos x = \frac{5}{4}$, then $\sin(2x) =$

(a) $\frac{9}{16}$ (b) $\frac{1}{5}$ (c) $\frac{5}{16}$ (d) $\frac{3}{4}$ (e) $\frac{7}{16}$

23. Suppose a regular octagon is drawn in the plane, and then each of the eight sides are extended in each direction to form eight lines. If the plane is cut up along these lines, how many of the resulting pieces have finite area? (Note: If this process is performed with a triangle, only one piece is finite.)

(a) 1 (b) 9 (c) 17 (d) 25 (e) 33

24. How many rational roots does the polynomial $2x^4 + 5x^3 + 4x^2 + 5x + 2$ have?

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

25. A right triangle has area 30 and perimeter 30. Find the length of the hypothenuse.

(a) 10 (b) $8\sqrt{2}$ (c) 12 (d) $9\sqrt{2}$ (e) 13

26. Suppose that the two large circles shown each have radius 1 and pass through the center of the other circle. A circle is inscribed in the overlap between the two larger circles. Find the area of the shaded region.

(a) $\frac{5\pi - 3\sqrt{3}}{12}$ (b) $\frac{7\pi - 3\sqrt{3}}{12}$ (c) $\frac{7\pi - 5\sqrt{3}}{12}$ (d) $\frac{5\pi - 6\sqrt{3}}{12}$ (e) $\frac{7\pi - 6\sqrt{3}}{12}$

27. Suppose $f(x) = x^3 + ax^2 + bx + c$ is a cubic polynomial such that the product of the roots is -3, the sum of the roots is 3, and $f(1) = 0$. What is $f(2)$?

(a) -19 (b) -3 (c) 0 (d) 3 (e) 19

28. Albert, Beth, and Carol were all born at 10:00 am on November 18, but in different years. When Alice was 3 years old, Beth was twice as old as Carol. When Beth was 8, Carol was three times as old as Alice. How old is Carol when Beth was eight times as old as Alice?

(a) 2 (b) 3 (c) 4 (d) 5 (e) 6

29. Suppose a trapezoid has diagonal lengths 5 and 7, and height 3. What is the area of the trapezoid?

(a) $\frac{3}{2}(\sqrt{40}+4)$ (b) $\frac{3}{2}(\sqrt{58}+4)$ (c) $\frac{3}{2}(\sqrt{58}+\sqrt{34})$ (d) 18 (e) $\frac{3}{2}(\sqrt{40}+\sqrt{34})$

30. Deb and Eric play a game in which the first player to make two moves in a row wins. If Deb has a 2/3 chance of making each move, what are her odds of winning?

(a) $\frac{5}{21}$ (b) $\frac{8}{21}$ (c) $\frac{10}{21}$ (d) $\frac{15}{21}$ (e) $\frac{16}{21}$

31. Consider an ellipse consisting of points x such that the sum of the distances of x to $A = (-1, 0)$ and $B = (3, 0)$ is equal to 5. What is the largest possible area of a quadrilateral $ACBD$ with C and D on the ellipse?

(a) $4\sqrt{21}$ (b) $5\sqrt{21}$ (c) $6\sqrt{21}$ (d) $7\sqrt{21}$ (e) $8\sqrt{21}$

32. Which of the following cannot be the sum of five consecutive positive integers?

(a) 10 (b) 15 (c) 20 (d) 25 (e) each of these represents such a sum

33. A city in the Midwest contains 2698 citizens. If at least x of the residents have the same two-letter initials, what is the smallest possible value of x?

(a) 2 (b) 3 (c) 4 (d) 5 (e) 6

34. What is the size of the largest set of integers between 10 and 40, any two of which are relatively prime?

(a) 8 (b) 9 (c) 10 (d) 11 (e) 12

35. Consider the "square Venn diagram" shown in the figure at right, with each square having area 1. If the areas of the four "overlap" regions are the same, what is the area of the shaded region?

(a) $\frac{1}{3}$ (b) $\frac{1}{4}$ (c) $\frac{1}{5}$ (d) $\frac{1}{6}$ (e) $\frac{1}{7}$

36. Which civilization is commonly attributed with developing the use of the number zero?

(a) the Mayans (b) the Europeans (c) the Greeks
(d) the Arabs (e) the Chinese

37. The concept of negative numbers is first seen in history

(a) before 0 A.D. (b) between 0 and 400 A.D. (c) between 400 and 800 A.D.
(d) between 800 and 1200 A.D. (e) after 1200 A.D.

38. Which of the following mathematicians is considered one of the most prolific in history?

(a) Pythagorus (b) Leonhard Euler (c) Blaise Pascal
(d) Sir Isaac Newton (e) Euclid

39. In what country did Blaise Pascal go up?

(a) Switzerland (b) Germany (c) Austria
(d) Italy (e) France

40. Which of the following was **not** developed by Blaise Pascal?

(a) barometer (b) hydraulic press (c) thermometer
(d) syringe (e) manual calculator